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NOTE

ON KERNELS IN PERFECT GRAPHS

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A kernel of a digraph D is a set of vertices which is both independent and absorbant. In 1983, C. Berge and P. Duchet conjectured that an undirected graph G is perfect if and only if the following condition is fulfilled: if D is an orientation of G (where pairs of opposite arcs are allowed) and if every clique of D has a kernel then D has a kernel. We prove here the conjecture for the complements of strongly perfect graphs and establish that a minimal counterexample to the conjecture is not a complete join of an independent set with another graph.

Topics on perfect graphs are of major interest in Graph Coloring Theory, mainly because of the famous Strong Perfect Graph Conjecture (see [1]). Another approach to perfectness was suggested in 1983 by Berge and Duchet [2]. A *kernel* of a directed graph D is a set K of vertices which is independent (i. e., no vertex in K is joined to another vertex in K) and absorbant (i. e. every vertex not in K has a successor in K). In digraphs considered here, pairs of opposite arcs (=directed cycles of lenght 2) are permitted.

Conjecture 1 (Berge, Duchet [2]).. *Let D be a directed graph whose underlying graph G is perfect. If every complete subgraph of G, directed by D, admits a kernel, then D itself has a kernel.*

Undefined terms concerning perfect graphs may be found in [1]. The problem of the existence of a kernel in a digraph is difficult [7, 10, 12] and this can explain why conjecture 1 is only settled for few special classes of perfect graphs, as i-trianguled graphs [14] or perfect line-graphs [13]. A weaker form of the conjecture is known for comparability graphs [13, 9], parity graphs [6, 11] and is settled for Meyniel graphs in a forthcoming paper [8]. See [3, 5, 13] for more information; a recent survey is [4]; whether conjecture 1 holds for comparability graphs or for parity graphs remains mysterious. Apart from perfect line-graphs, any graph G of one of the classes quoted above is *strongly perfect*, i. e. every induced subgraph G' of G possesses an independent set which meets every maximal clique of G'. Here we prove that conjecture 1 holds for the graphs whose complement is strongly perfect

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(*Theorem 3*). In particular the conjecture is true for the complements of perfectly orderable graphs and for the complements of Meyniel graphs since those graphs are known to be strongly perfect (Chvátal [1]; Ravindra [1]).

In the sequel, the term *clique* is used as well for the sets of vertices of any complete subgraph or subdigraph (not necessarily maximal) and for the complete graph (or digraph) the clique induces. A kernel of a (directed) clique is nothing else but a *receiver*, i. e. a vertex which is a successor of every other vertex of the clique. On the contrary a vertex of whom every other vertex is a successor is named an *emitter*. A directed graph D in which every clique admits a receiver is called a *admissible direction* of its underlying undirected graph. An undirected graph G is said to be *solvable* [3] or *nearly-perfect* [2] if every admissible direction of G has a kernel. The results of this paper are based on the following elementary lemma:

Lemma 2. *Suppose D is an admissible direction; then every clique of D contains an emitter. Consequently, the reversal digraph D^{-1} is also an admissible direction.*

Theorem 3. *The complement of any strongly perfect graph is solvable.*

Proof. By induction on the number of vertices. The theorem is trivial for small graphs. Let D be an admissible direction of a graph G of order ≥ 2 with strongly perfect complement and assume the theorem holds for smaller graphs. We have to show that D has a kernel. Since $\overline{G}$ is strongly perfect, the graph G contains a clique C which meets every maximal independent set of G. By Lemma 2, the clique C, directed by D, admits an emitter a. The digraph $D-a$ is an admissible direction of $G-a$ and the complement of $G-a$ is strongly perfect. By the induction hypothesis, the digraph $D-a$ possesses a kernel K.

If K meets $C-a$, then K is a kernel of D. Otherwise, since C meets every maximal independent set of G, the set K is included in a larger independent set $K\cup\{b\}$ of G. Since K is absorbant in $D-a$, we have $a=b$ and $K\cup\{a\}$ is a kernel of D. ∎

It would be interesting to obtain an analogue of Lovász's Theorem on perfect graphs [1]. *Is the complement of a solvable graphs also solvable* ? To date, it is not even known whether the complete join of two disjoint solvable graphs is solvable, although the complete join operator corresponds to the disjoint union in the complement graphs (disjoint union obviously preserves solvability). We have the following partial result.

Proposition 4. *The complete join of a solvable graph with an independent set is solvable.*

Proof. Let G' be a solvable graph on vertex set V and let S be an independent set disjoint from V. Let us consider an admissible direction D of the graph G, defined as the complete join of G' with S. We have to find a kernel of D. We may assume that S is not a kernel of D, thus V contains a vertex a with no successor in S. Necessarily, a is a successor of all elements of S. We denote by A the set of vertices of V that are successors of all elements of S and by B the set $V-A$. To find a kernel of D, if suffices to show that the induced subdigraph $D[V]$ possesses a kernel which meets A.

Suppose a has a successor b in B. By the definition of A, there exists $s\in S$ such that (b,s) is an arc of D, but (s,b) is not. Moreover, (a,s) is not arc of D, hence a is the only possible receiver os the clique $\{a,b,s\}$ of D. It follows that the arc$e=(b,a)$ must be present in D. Let $D(b)$ denote the subdigraph of D obtained by the deletion of arc(a,b). We claim $D(b)$ *is again an admissible direction of* G. Otherwise, some clique C of G has no emitter in $D(b)$ but has one in D. Hence $a,b\in C$ and a is the only emitter of C in D. This forces $C\cap S=0$, but then $C\cup\{s\}$ is a clique of D with no emitter. Contradiction.

We can iterate the same argument, after successive removals of all arcs of the form (a,b) with $b\in B$, we obtain a subdigraph D_0 of D with both properties:

(i) D_O is an admissible direction of G;

(ii) In D_0, no successor of a is in B.

Now, since G' is solvable, the induced digraph $D_0[V]$ has a kernel K. Either $a\in K$ or a has a successor in K. In both cases K meets A and constitutes a kernel of D. ∎

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